RESULT ON PSEUDO – HAMILTONIAN COMPLETE GRAPHS

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ABSTRACT

In 1856, Hamiltonian introduced the Hamiltonian Graph where a Graph which is covered all the vertices without repetition and end with starting vertex. In this Paper I would like to prove Result on pseudo – Hamiltonian Complete Graphs.

\textbf{Keywords:} Graph, Hamiltonian Graph, Complete Graph, Neighborhood, Locally Complete Graph.
INTRODUCTION:

Given a graph $G$ and a positive integer $k$, denote by $G[k]$ the graph obtained from $G$ by replacing each vertex of $G$ with an independent set of size $k$. A graph $G$ is called pseudo-$k$ Hamiltonian-complete if $G[k]$ is Hamiltonian-complete, i.e., every two distinct vertices of $G[k]$ are complete by a Hamiltonian path.

A graph $G$ is called pseudo Hamiltonian-complete if it is pseudo-$k$ Hamiltonian-complete for some positive integer $k$.

This paper proves that a graph $G$ is pseudo-Hamiltonian-complete if and only if for every non-empty proper subset $X$ of $V(G)$, $|NG(X)| > |X|$.

Given a graph $G$ and a positive integer $k$, denote by $G[k]$ the graph obtained from $G$ by replacing each vertex of $G$ with an independent set of size $k$.

To be precise, $G[k]$ has vertex set $\{v_i : v \in V(G); i=1,2,...,k\}$, two vertices $v_i$ and $u_j$ are adjacent if and only if $vu$ is an edge of $G$.

A graph $G$ is called pseudo-$k$ Hamiltonian-complete if $G[k]$ is Hamiltonian-complete, i.e., every two distinct vertices of $G[k]$ are complete by a Hamiltonian path. Suppose $G$ is a graph and $x$ and $y$ are vertices of $G$.

An $x$-$y$ walk $W$ of $G$ is called a regular Hamiltonian walk if there is a positive integer $k$ such that each vertex of $V(G)$ occurs exactly $k$ times in $W$. It is easy to see that if $G$ is pseudo-Hamiltonian-complete, then for every pair of distinct vertices $x$ and $y$ of $G$ there exists an $x$-$y$ regular Hamiltonian walk.

An $x$-$y$ walk $W$ is called a pseudo-edge if there is an integer $k>0$ such that each vertex of $V(G)$ – $\{x; y\}$ occurs $k$ times in $W$, and each of $x$ and $y$ occurs $(k + 1)$ times in $W$.

We are interested in graphs for which every pair of distinct vertices is complete by a pseudo-edge. Given a pseudo-Hamiltonian-complete graph $G$, we denote by $p(G)$ the minimum number $k$ for which $G[k]$ is Hamiltonian complete. we prove that any pseudo-Hamiltonian-connected graph with a Hamiltonian cycle is pseudo-$2$ Hamiltonian-complete.

1.1 Definition: A graph – usually denoted $G(V,E)$ or $G = (V,E)$ – consists of set of vertices $V$ together with a set of edges $E$. The number of vertices in a graph is usually denoted $n$ while the number of edges is usually denoted $m$.

1.2 Definition: Vertices are also known as nodes, points and (in social networks) as actors, agents or players.

1.3 Definition: Edges are also known as lines and (in social networks) as ties or links. An edge $e = (u,v)$ is defined by the unordered pair of vertices that serve as its end points.

1.4 Example: The graph depicted in Figure 1 has vertex set $V = \{a,b,c,d,e,f\}$ and edge set $E = \{(a,b),(b,c),(c,d),(c,e),(d,e),(e,f)\}$.

![Figure 1](image)

1.5 Definition: Two vertices $u$ and $v$ are adjacent if there exists an edge $(u,v)$ that connects them.

1.6 Definition: An edge $(u,v)$ is said to be incident upon nodes $u$ and $v$.

1.7 Definition: An edge $e = (u,u)$ that links a vertex to itself is known as a self-loop or reflexive tie.

1.8 Definition: Every graph has associated with it an adjacency matrix, which is a binary $n \times n$ matrix $A$ in which $a_{ij} = 1$ and $a_{ji} = 1$ if vertex $v_i$ is adjacent to vertex $v_j$, and $a_{ij} = 0$ and $a_{ji} = 0$ otherwise. The natural graphical representation of an adjacency matrix is a table, such as shown below.

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<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
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<th>e</th>
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</thead>
<tbody>
<tr>
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<td>1</td>
<td>0</td>
<td>0</td>
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<td>b</td>
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<td>f</td>
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<td>1</td>
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</tbody>
</table>

Adjacency matrix for graph in Figure 1.

1.9 Definition: Examining either Figure 1 or given adjacency Matrix, we can see that not every vertex is adjacent to every other. A graph in which all vertices are adjacent to all others is said to be complete.

1.10 Definition: While not every vertex in the graph in Figure 1 is adjacent, one can construct a sequence of adjacent vertices from any vertex to any other. Graphs with this property are called connected.
1.11 Note: Reachability. Similarly, any pair of vertices in which one vertex can reach the other via a sequence of adjacent vertices is called reachable. If we determine reachability for every pair of vertices, we can construct a reachability matrix $R$ such as depicted in Figure 2. The matrix $R$ can be thought of as the result of applying transitive closure to the adjacency matrix $A$.

![Figure 2](image)

1.12 Definition: A walk is closed if $v_o = v_n$. degree of the vertex and is denoted $d(v)$.

1.13 Definition: A tree is a connected graph that contains no cycles. In a tree, every pair of points is connected by a unique path. That is, there is only one way to get from $A$ to $B$.

![Figure 3](image)

1.14 Definition: A spanning tree for a graph $G$ is a sub-graph of $G$ which is a tree that includes every vertex of $G$.

1.15 Definition: The length of a walk (and therefore a path or trail) is defined as the number of edges it contains. For example, in Figure 3, the path $a,b,c,d,e$ has length 4.

1.16 Definition: The number of vertices adjacent to a given vertex is called the degree of the vertex and is denoted $d(v)$.

1.17 Definition: An Eulerian circuit in a graph $G$ is circuit which includes every vertex and every edge of $G$. It may pass through a vertex more than once, but because it is a circuit it traverse each edge exactly once. A graph which has an Eulerian circuit is called an Eulerian graph. An Eulerian path in a graph $G$ is a walk which passes through every vertex of $G$ and which traverses each edge of $G$ exactly once.

1.18 Example: Königsberg bridge problem: The city of Königsberg (now Kaliningrad) had seven bridges on the Pregel River. People were wondering whether it would be possible to take a walk through the city passing exactly once on each bridge. Euler built the representative graph, observed that it had vertices of odd degree, and proved that this made such a walk impossible. Does there exist a walk crossing each of the seven bridges of Königsberg exactly once?

Figure 4: Königsberg problem

HAMILTONIAN GRAPHS, COMPLETE GRAPHS AND PSEUDO-HAMILTONIAN – COMPLETE

2.1 Definition: A Hamilton circuit is a path that visits every vertex in the graph exactly once and return to the starting vertex. Determining whether such paths or circuits exist is an NP-complete problem. In the diagram below, an example Hamilton Circuit would be AEFGCDBA.

2.2 Example:

Figure: Hamilton Circuit would be AEFGCDBA.

2.3 Definition: Compete Graph: A simple graph in which there exists an edge between every pair of vertices is called a complete graph.

2.4 Theorem: Suppose G is pseudo-Hamiltonian - Complete. If G has a Hamiltonian cycle then p(G) ≤ 2.

Proof. Suppose G is pseudo-Hamiltonian- Complete and has a Hamiltonian cycle.

By Known theorem, for every proper subset X of V(G), |NG(X)| > |X|.

For any (not necessarily distinct) vertices x and y of G, let H(x; y) be the sub-multi graph of G defined as in the proof of known theorem.

Namely, when x ≠ y, each of x and y has degree 1 and every other vertex has degree 2 in H(x; y); when x = y, then x has degree 0 and every other vertex has degree 2 in H(x; x).
Let $C$ be a Hamiltonian cycle of $G$.

Then $H = C \cup H(x; y)$ is a complete sub-multigraph of $G$ such that when $x \neq y$ then $d_H(x) = d_H(y) = 3$ and $d_H(u) = 4$ for $u \neq x, y$;

and

when $x = y$ then $d_H(x) = 2$ and $d_H(u) = 4$ for $u \neq x$.

Thus, $H$ has an Eulerian trail connecting $x$ and $y$, which is then an $x$- $y$ regular Hamiltonian walk of $G$ which traverses each vertex of $G$ exactly twice.

The toughness $t(G)$ of a graph $G$ is defined as

$$t(G) = \min\{|S| / k(G - S) : S \text{ is a vertex cut set of } G\}$$

where $k(G - S)$ is the number of components of $G - S$.

It was conjectured by Chvatal [3] that there is a real number $r_0$ such that any graph $G$ with $t(G) > r_0$ is Hamiltonian.

Chvatal also conjectured that letting $r_0 = 2$ would be enough.

Note that non-Hamiltonian graphs exist with toughness at least $t$ for each $t < 2$.

While the first conjecture remaining open, the second one is recently disproved by Bauer, Broersma and Veldman.

**Conjecture 1.** There is a real number $r_0 > 2$ such that for any graph $G$; if $t(G) > r_0$ then $G$ is pseudo-2 Hamiltonian-connected.

**Conjecture 2.** There is a real number $r_0$ and an integer $k$ such that for any graph $G$; if $t(G) > r_0$ then $G$ is pseudo-$k$ Hamiltonian-connected.

**References :**


